"I see it, but I don't believe it!"

— Georg Cantor, on discovering that the set of
points on the interval [0, 1] is equivalent to
the set of points in p-dimensional space

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Georg Ferdinand Ludwig Phillip Cantor, man of mystery, was one of few modern mathematicians to be credited with single-handedly developing rigorously an important foundation of mathematics, set theory.

Initially developed as a tool in his other workings, Cantor was the first person to even conceive of set theory as its own branch of mathematics, a branch that has proved crucial in many other areas of theory. One notion in particular which Cantor developed was the existence of ‘transfinite’ numbers-- those which were not quite infinite, not beyond all description and comprehension, but were greater than any finite number. In this paper, we will attempt to give an overview of Cantor’s theories of transfinite numbers, starting with basics of set theory, including cardinality and ordinality of sets, as a foundation.

This paper also includes an appendix of several proofs referenced in this paper. We felt such proofs were not relevant enough to our focus to warrant inclusion in the body of the paper, but that it was still necessary for us to demonstrate understanding of these concepts, as well as allow for the possibility of a reader seeking some slightly more in-depth explanations of these claims.

Basics of Set Theory

Cantor begins his theory of transfinites by laying a foundation in set theory. The basic units of set theory are elements, which are in turn collected in sets, or classes, which Cantor called aggregates. An aggregate M (from the German *Menge*), made up of a number of objects m, can be written as

(1) $M=\{m\}$.

Sets can be manipulated in several ways. First, they can be combined with one another to form union sets: for the sets M, N, P, and so on, the union of all these is (M, N, P, …). The inverse of this operation is the taking of a subset, M_1, from the set M. All of the elements of M_1 are also elements of M.

One of the most important notions in set theory is that of the equivalence of sets. Two sets M and N can be said to be equivalent to one another if there can be drawn a one-to-one correspondence between their elements, that is, “they can be put in such a relation to one another that to every element of each one of them corresponds one and only one element of the other" (Cantor, pg. 86). We express this with the symbols

(2) $M \sim N$ or $N \sim M$

This allows for several more laws regarding sets. A set is always equivalent to itself.

(3) $M \sim M$

Also, two sets which are equivalent to a third are also equivalent to each other.

(4) from $M \sim P$ and $P \sim N$ follows $M \sim N$

If M, N, and P are each equivalent to M', N', P', then (M, N, P, …) is equivalent to (M', N', P', …). Equivalency can be well illustrated with the classic example of two lines AB > CD. If lines are drawn through AC and BD, they intersect at a point O. Now any line drawn from O through CD, to AB, denotes a one-to-one correspondence between the point at which it crosses AB and the point at which it crosses CD.

Most sets which we deal with in reality are finite sets. For example, the set of all atoms in a molecule, or the set of all students in a class. However, there are also sets which have an infinite number of elements, such as the set of all natural numbers, or the set all points on a line segment. These sets are defined as infinite by saying that there can be drawn a one-to-one correspondence (equivalence) between the entire set, and a subset of itself, a definition formulated by the mathematician Bolzano. This can be illustrated using the same model used above in demonstrating one-to-one correspondence in general. Now, define a third line EF < AB, which is co-linear with AB. Projecting each point on CD to a point on EF using the same procedure proves that the points on a line segment AB are the members of an infinite set.

From AB ~ CD and CD ~ EF follows that AB ~ EF, thus AB is equivalent to a subset of itself.

Another example of this is the phenomenon noticed by Galileio in his 1638 work *Two New Sciences*. Namely, that the set of natural numbers can be put in a one-to-one correspondence with the set of squares, a subset of the set of natural numbers, i.e. {1, 2, 3, …} corresponds to {1, 4, 9, …}.

We can also compare sets to one another. To say that a set B is greater than a set A, that is, B > A, is to say that there exists a one-to-one correspondence between A and a subset of B, but no correspondence between A and B. Simply, there are more elements in B than in A. This concept seems elementary for our familiar finite sets, but the definition becomes a powerful tool when we compare infinite sets, which will come later.

It is also true that infinite sets cannot be made finite by the exclusion of elements. This is derived from the assertion that a set A, when the element x is removed from it, continues to be an infinite set[1].

A crucial concept in set theory is that of a set being well-ordered. Any well-ordered set is called a sequence. By this we mean “any well-defined aggregate whose elements have a given definite succession such that there is a *first* element, a definite element follows every on (if it is not the last), and to any finite or infinite aggregate a definite element belongs which is the *next* following element in the succession to them all (unless there are no following elements in the succession)” (Cantor, 61).

Well-ordering is a subset of simple-ordering. Simple ordering does not require a first element. For example, the set of rational numbers, when in order of their magnitude, is simply-ordered but not well ordered, for between any who elements in the set, another can be found. However, when each rational number p/q is ordered by the magnitude of $p+q$, the sequence is well-ordered as well. (When using this rule for ordering, the sequence will contain subsets of elements whose sums are equal. In this case, we order each subset by the actual magnitude of each element.)

An important operation (as we shall see later on) which can be performed on a set is the derivation of its power set. For any set A, the power set of A, P(A), is the set of all the subsets of A. So for some set A, where A = {1, 2, 3},

P(A) = {Ø, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

where Ø signifies the null, or empty, set. For a set with *n* elements, the power set will have 2^n elements. It should also be noted that regardless of whether A is finite or not, the power set of A will be greater than A.

In this discussion of basic set theory, we have covered enough background to introduce Cantor's concept of cardinality.

Cardinality

Given a set of objects, both the specific elements of the set and their order are important. However, according to Cantor, when we, "by means of our active faculty of thought,…make abstraction of the nature of the various elements *m* and the order in which they are given," the result is the cardinal number, or power, of the set. For finite sets, the cardinality of the set is simply the number of elements in the set. For example, the cardinality of the set of natural numbers less than *n*, i.e. {1, 2, 3, …, *n*}, is equal to *n*. If two sets are equivalent, such as the sets {1, 2, 3} and {1, 4, 9}, their cardinals are equal.

A cardinal number, in addition to representing a single property, can be thought of as a set of units, which can be mapped in a one-to-one correspondence to the elements of the set from which it is derived. Therefore, a set A is equivalent to it's cardinality: $A \sim \overline{\overline{A}}$ and $\overline{\overline{A}} \sim A$, where cardinality is denoted by the double bars. Also, a cardinal number can be thought of as the set of all sets which share a single cardinality.

Any set which is finite or which has the same cardinality as the natural numbers can be referred to as countable or, to use Cantor's term, denumerable. In accordance with this notion, if A and B are two denumerable sets, then the union of two sets A ⋃ B (also written "(A, B)") is also denumerable. Cantor also developed a proof that the set of all rational numbers (that is, numbers that can be written as fractions) is equivalent to the set of natural numbers[2].

Just as the sizes of real numbers can be compared, so too can the sizes of cardinals. The simplest comparison is that of equality. Given two sets A and B, $A \sim \overline{\overline{A}}$ and $B \sim \overline{\overline{B}}$. Therefore, if A ~ B, $\overline{\overline{A}} = \overline{\overline{B}}$.

Proving the inequality of two cardinals is slightly more involved. Essentially, consider two sets M and N in which (1) no subset of M is equivalent to N and (2) there *is* a subset of N which is equivalent to M. Under these two conditions, the cardinality of $\overline{\overline{M}}$ will be less than the cardinality of $\overline{\overline{N}}$. (In a finite example, the number of elements in M would be less than the number of elements in N.) If the conditions are reversed, then $\overline{\overline{N}}$ would be less than $\overline{\overline{M}}$. Thus, no cardinal can be both less than *and* greater than another. Cantor states that with any two cardinal numbers $\overline{\overline{M}}$ and $\overline{\overline{N}}$, one and only one of the following conditions can hold:

(1) $\overline{\overline{M}} = \overline{\overline{N}}$ (2) $\overline{\overline{M}} > \overline{\overline{N}}$ (3) $\overline{\overline{M}} < \overline{\overline{N}}$.

It is also true that if one cardinal is greater than another, it will also be greater than a subset of the other.

Operations on Cardinal Numbers

Cardinals can be added, multiplied, and exponentiated, just as real numbers can, but in different ways. The sum of the cardinals of two sets is equal to the cardinality of the union of the sets. This follows from the fact that if M ~ M’ and N ~ N’, then (M, N) ~ (M’, N’). Because the order of elements in the sets does not matter, addition of cardinals is commutative, that is, if $\overline{\overline{M}} = a$ and $\overline{\overline{N}} = b$,

(5) $a + b = b + a$

Addition is also associative, so for any three cardinal numbers $a, b, c,$ we have

(6) $a + (b + c) = (a + b) + c$

Multiplication of cardinals is somewhat more complicated. “Any element m of an aggregate M can be thought to be bound up with any element n of another aggregate N so as to form a new element (m, n); we denote by (M.N) the aggregate of all these bindings (m, n), and call it the 'aggregate of bindings of M and N.' Thus, (M.N) = {(m, n)}" (Cantor, pg. 92). That is to say, (M.N) symbolizes the set of all the pairs (m, n), which are the corresponding elements in M and N. As the power of (M.N) depends only on the powers $\overline{\overline{M}} = a$ and $\overline{\overline{N}} = b$, we can define the product of a.b as

(7) $a.b = \overline{\overline{(M.N)}}$

Distributivity, commutativity, and associativity all hold for multiplication of cardinals as well, that is,

(8) $a.b = b.a$ since $(M.N) \sim (N.M)$

(9) $a.(b.c) = (a.b).c$ since $(M.(N.P)) \sim ((M.N).P)$

(10) $a(b+c) = ab + ac$ since $(M.(N, P)) \sim ((M.N), (M.P))$

Cantor defines an operation called "covering", in which a specific element m_0 of the set M is "bound up" with every element n of the set N. The function by which m_0 is connected with every n is a one-value (monotonous) function:

$$f(n) = m_0$$

Evidentially, the collection of all the mappings between N and M is a set whose elements are defined by f(N). This so-called "covering-aggregate" is denoted as follows:

$$(N \mid M) = \{f(N)\}$$

Note that unlike (N . M), (N | M) is not commutative.

Given two other sets M' ~ M and N' ~ N, the covering-aggregate of these is the same as that of M and N:

$$(N \mid M) \sim (N' \mid M')$$

By definition, the cardinality of the covering-aggregate of (N | M) depends only on the cardinality of N and M. This is the definition of exponentiation of cardinalities:

$$\overline{\overline{M}}^{\overline{\overline{N}}} = \overline{\overline{(N \mid M)}}$$

Following are three theorems which demonstrate that the truth of three properties of exponentiation hold true for cardinals:

$((N \mid M) . (P \mid M)) \sim ((N, P) \mid M)$ $\mathbf{a}^{\mathbf{b}} . \mathbf{a}^{\mathbf{c}} = \mathbf{a}^{\mathbf{b+c}}$

$((P \mid M) . (P \mid N)) \sim (P \mid (M . N))$ $\mathbf{a}^{\mathbf{c}} . \mathbf{b}^{\mathbf{c}} = (\mathbf{a} . \mathbf{b})^{\mathbf{c}}$

$(P \mid (N \mid M)) \sim ((P . N) \mid M)$ $(\mathbf{ab})^{c} = \mathbf{a}^{\mathbf{b} \cdot \mathbf{c}}$

Greater and Lesser Infinities

Earlier in the paper, a proof was given that the set of all rational numbers is equivalent to the set of all real numbers. This was proven by showing that the rationals can be well ordered, that is, made countable, and so equivalent to the set of natural numbers. Therefore, both of these sets have the same cardinal number. As these are both infinite sets, no known number can represent their cardinality. Their cardinality is what Cantor called transfinite, a name he justified by proving that $\overline{\overline{N}}$ was greater than any finite number, but also simply the smallest of the transfinite numbers, and therefore not absolutely infinite. This concept will be addressed more later in the paper. Cantor chose to represent this with the first letter of the Hebrew alphabet, À$_0$ (aleph sub naught). So we now have

$$\overline{\overline{Q}} = \overline{\overline{N}} = \grave{A}_0$$

Both of these sets are infinite, so it might seem to follow that the cardinality of any infinite set would be À$_0$. Strange as it sounds, this is not true. A prime example is the set of all real numbers, R. Cantor proved that it is not possible to set up a one-to-one correspondence between R and N, even though they are both infinite[3].

Cantor called the cardinality of the real numbers C, for continuum. So we now have the set of natural numbers, which can be put into one-to-one correspondence of a subset of R, but not so with all of R. Here we have just stated the exact definition of one set being greater than another. So regardless of the fact that N and R are both infinite, we have:

$$R > N$$

Finite and Transfinite Ordinals

An ordinal, like a cardinal, describes over-all properties of a set, and at the same time denotes a collection of all sets among which this property is the same. However, while cardinals describe only the "size" of a set, ordinals describe both the "size and shape." Simply, order counts.

Ordinals can be thought of as being a type to which sets belong. Only simply-ordered sets, a type of sequence, have ordinality.

There are several types of ordinals, each denoted by a specific symbol, and representing a class of sequences of a given "shape". All ordinals of the same type are said to be ordinally similar to each other.

The first common ordinal type is the set of all sequences which have both definite first and last elements. All of these finite sequences have an ordinal type equal to their cardinality. That is, the ordinality of a finite sequence is equal to the number of items it has.

The rest of the common types deal with sequences which, in one way or another, do not terminate. The second type denotes the set of all "progressions", all sequences which have a definite first element but have no definite last element. These are denoted by the ordinal ω. An example of a series of this type is the set of all integers, $Z = \{1, 2, 3, \ldots\}$. The next type of ordinal is $*\omega$. This type encompasses all series which have a definite last element, but have no definite first element, such as $\{\ldots, -3, -2, -1, 0\}$. These types are called "regressions." The last type in this discussion sequences have the ordinal type $*\omega + \omega$. This is the type which denotes the set of all "unlimited" sequences, sequences which have neither a definite first or last element (but are still well-ordered!). This makes sense, as any unlimited sequence, if an "origin" is chosen to separate the sequence into two subsequences, the first subsequence will be a regression and the second a progression. For example the set of all natural numbers, $N = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ can be split into the regression $\{\ldots, -3, -2, -1, 0\}$ and the integers $Z = \{1, 2, 3, \ldots\}$.

Operations on Ordinals

These basic building blocks can now be used to describe sequences with more complicated shapes. While the notation used involves the standard symbols for addition, multiplication, and exponentiation, these are not the operations that are being performed. (For example, the "+" is more like a concatenation symbol than an addition symbol.) We begin with a progression such as the following. This sequence has ordinality ω.

$$\{1, 2, 3, 4, 5, 6, \ldots\}$$

If we were to add a finite sequence, such as {42, 23}, to the beginning of the sequence, the sequence would look like this:

$$\{42, 23, 1, 2, 3, 4, 5, 6, \ldots\}$$

The order is no longer ascending, but it is still a progression. Thus, the "addition" of a finite sequence to the beginning of a progression does nothing to change it's shape (just as adding elements to a set equivalent to the set of all natural numbers does not change its cardinality): $2+\omega = \omega$.

However, consider the case of adding the same finite sequence to the *end* of a progression:

$$\{1, 2, 3, 4, 5, 6, \ldots, 42, 23\}$$

In this case the shape has changed and $\omega+2$ is not equal to ω. "Thus, as one can see here, everything depends on the position of the finite vis-à-vis the infinite. If the former comes first, then it merges with the infinite and disappears in it; if, however, if *contents* itself and takes its place behind the infinite, then it is preserved and joins with the former to become a new, because modified, infinite" (Cantor, Foundations of General Theory of Manifolds, pg. 77).

Multiplication operates in a similarly non-commutative fashion. The sequence "$a_1, a_2, a_3, a_4, \ldots$" (which has ordinality ω) can be doubled in two ways. First, each element can be doubled, as in: "$a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \ldots$". This new sequence has the same shape as the original sequence, that is, it is a progression. Therefore, it has the same ordinality. Thus, $2 . \omega = \omega$. The other method of doubling produces different results: "$a_1, a_2, a_3, a_4, \ldots, b_1, b_2, b_3, b_4, \ldots$", with the ordinality $\omega . 2$. This sequence has a different, more complicated shape, and thus $\omega . 2$ does not equal ω.

About exponentiation: in the example of multiplication of the form $\omega . 2$, we doubled the number of progressions in the set, placing the second after the first. Exponentiation is an extension of this concept. For a progression *s*, a sequence with ordinality ω, a new sequence with ordinality ω^n will be in this form:

$$s', s'', s''', s^{(IV)}, \ldots, s^{(n)}$$

And a new sequence with ordinality ω^ω will be of the form:

$$s', s'', s''', s^{(IV)}, \ldots, s^{(v)}, \ldots$$

That is, a sequence which contains an "infinite" number of "infinite" sequences.

Transfinite Cardinals

According to Cantor, "aggregates with finite cardinal numbers are called 'finite aggregates,' all others we will call 'transfinite aggregates' and their cardinal numbers 'transfinite cardinal numbers.'" (Cantor, 103)

The smallest transfinite cardinal number is $\aleph_0$. This number is the cardinality of the set of all *finite cardinal numbers*, called *v*. For example:

Finite cardinal numbers such as $\overline{\overline{\{1,2,3\}}} = \overline{\overline{\{1,4,9\}}} = \overline{\overline{\{7,42,90\}}}$ and $\overline{\overline{\{1,2,3,4\}}} = \overline{\overline{\{1,4,9,16\}}}$ (and so on) can be collected into *v*, all the finite cardinal numbers: 1,2,3,4,..

The cardinality of {*v*}, the set of all *v*, is written $\overline{\overline{\{v\}}}$, and is equal to $\aleph_0$.

It can be shown that $\aleph_0$ is not equal to any finite number μ:

Given the aggregate {*v*}, a new aggregate ({*v*}, e_0) will be equivalent to the original: $\{v\} \sim (\{v\}, e_0)$. This is because we can map element e_0 of the second set to the first element of the first set, and so on, until we get to element *v* of the second set, which we map to element *v*+1 of the first set. Thus, the two sets have a one-to-one correspondence between them. According to the rule for the addition of cardinals (detailed in the next section), then, $\aleph_0 + 1 = \aleph_0$. However, for any μ, μ + 1 is discrete from (not equal to) μ. Therefore $\aleph_0$ is not among the finite numbers.

It can also be shown that $\aleph_0$ is *greater than* any finite number. This is derived from three facts: **(1)** $\mu = \overline{\overline{(1,\ 2,\ 3,\ \ ,\mu)}}$, **(2)** no subset of (1, 2, 3, … , μ) is equivalent to {*v*}, **(3)** (1, 2, 3, … , μ) *is* a subset of {*v*}. However, the lemmas on which this proof rests are beyond the scope of this paper.

We now bring in the concept of a power set, which we have already defined as the set of all possible subsets of a given set. It was also noted that a for a set A with *n* elements, the power set will have 2^n elements. Expressed in terms of cardinality:

$$P(A) = 2^{\bar{\bar{A}}}$$

Cantor referred to this power set as numbers of the second class. And as the cardinality of the natural numbers is À_0, so is the cardinality of $P(N) = 2^{\bar{\bar{\text{À}}}_0}$, which cantor called À_1. The power set of that has a cardinality of À_2, which is greater than À_1, a process which can be repeated indefinitely, so we now have

$$\text{À}_0 < \text{À}_1 < \text{À}_2 < \text{À}_3 \ldots$$

Each set is infinite, but is also strictly smaller than the next.

Of course C, the cardinality of the real numbers, must fit in here somewhere. All we know so far, however, is that it is greater than À_0. We know nothing of its relation to À_1 or any of the other transfinite cardinals. Cantor hypothesized about the Continuum Hypothesis:

$$\text{À}_1 = C$$

but was never able to prove it.

Operations on Transfinite Cardinals

Operations on transfinite cardinals may seem bizarre at first, but in many cases is simply an articulation of a concept which already makes sense. With addition, for example, we find that:

$$\text{À}_0 + \text{À}_0 = \text{À}_0$$

What this means is only that adding one countable set to another yields a countable set. This same equation can also be written

$$\text{À}_0.2 = \text{À}_0$$

By adding À_0 repeatedly to both sides, we find that

$$\text{À}_0.n = n.\text{À}_0 = \text{À}_0$$

Eventually, we find ourselves with the result

$$\text{À}_0 . \text{À}_0 = \text{À}_0$$

Cantor provides a more rigorous proof of this result. However, this is beyond the scope of this paper. Now By taking $\text{À}_0 . \text{À}_0 = \text{À}_0$, or $\text{À}_0^2 = \text{À}_0$, and multiplying both sides of the equation by À_0, it follows that

$$\text{À}_0^3 = \text{À}_0^2 = \text{À}_0$$

Which of course leads to the statement that for any number n,

$$\text{À}_0^n = \text{À}_0$$

Conclusion

While mathematicians before Cantor had been able to talk about degrees of infinity in vague terms, he was the first to be able to describe so precisely a working system of relative, definite, “proper” infinites. As mathematicians after him have agreed, the notion and theory of the transfinites is essential to the progression of set theory, function theory, and the work built on their foundation.

Cantor, as he had hoped, found transfinites exceedingly helpful in his exploration of set theory: “it would hardly be possible for me to make without constraint the least step forward in the theory of aggregates”, although he too was wary of the notions at first: “I was logically forced, almost against my will, because in opposition to traditions which had become valued by me in the course of scientific researches extending over many years, to the thought of considering the infinitely great, not merely in the form of the unlimitedly increasing, and in the form, closely connected with this, of convergent infinite series, but also to fix it mathematically by numbers in the definite form of a completed infinite.”

However, Cantor’s theories also caused great controversy as paradoxes were discovered in the system. One well-known paradox, Russell’s paradox, is as follows:

Consider a set *w* defined as follows:

$$w = \{x : x \quad x\}$$

If any *w* is a member of *w* then is cannot be a member of *w*.

If any *w* is *not* a member of *w*, then is a member of *w*.

This is an illustration of the problems with Cantor's willingness to consider "the cardinal number of all things" (Cantor, 99). The paradoxes which arose in Cantor's work were some of the primary contributions to the differences between two of the major schools of mathematics — the intuitionists and the formalists.

The intuitionists viewed Cantor's theories with much distrust. To them, transfinites, along with irrational numbers, were a non-intuitive concept, not apparent in nature, and could be eliminated altogether from mathematics.

David Hilbert, founder of the school of formalism, believed that the foundation on which set theory and Cantor's ideas were built was sound, and inconsistencies could be resolved by increasing the precision with which ideas were expressed. For example, Russell's paradox can be easily resolved if one considers a set which references itself, sometimes called a "self-swallowing set," to be not a set at all but a meta-set.

In his 1925 speech to the congress of the Westphalian Mathematical Society, he gives a plan for eliminating contradictions from mathematics, a battle cry which we feel summarizes the spirit of this paper:

"Wherever there is any hope of salvage, we will carefully investigate fruitful definitions and deductive methods. We will nurse them, strengthen them, and make them useful. No one shall drive us out of the paradise that Cantor has created for us."

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